

Neutrino oscillations in a stochastic model for space-time foam

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We study decoherence models for flavour oscillations in four-dimensional stochastically fluctuating space times and discuss briefly the sensitivity of current neutrino experiments to such models. We pay emphasis on demonstrating the model dependence of the associated decoherence-induced damping coefficients in front of the oscillatory terms in the respective transition probabilities between flavours. Within the context of specific models of foam, involving point-like D-branes and leading to decoherence-induced damping which is inversely proportional to the neutrino energies, we also argue that future limits on the relevant decoherence parameters coming from TeV astrophysical neutrinos, to be observed in ICE-CUBE, are not far from theoretically expected values with Planck mass suppression. Ultra high energy neutrinos from Gamma Ray Bursts at cosmological distances can also exhibit in principle sensitivity to such effects.

I. INTRODUCTION

The possibility that quantum gravity involve models with stochastic fluctuations of the associated metric field, around some fixed background value, say flat Minkowski space time, may not be an unrealistic one. Such stochastic models of gravity lead to observable consequences in principle, ranging from light cone fluctuations [1] to induced decoherence for matter propagating in such fluctuating space times [2]. Space time foam models naturally include such stochastic space time backgrounds.

We describe here briefly the possible appearance of a stochastic space-time background by the D-particle foam model. It should be remarked that foam models do not consist exclusively of microscopic black holes, although this is the most popular model. The authors of [3] have provided another type of foam, inspired by brane-world scenarios. According to this model of *D-particle foam*, our brane world moves in a bulk space, punctured by point-like D-particles, which are point-like defects in membrane theory, characterised though by an infinity of super-Planckian string states. During such a motion, D-particles from the bulk cross our brane world, and interact with propagating matter in it, represented by open strings with their end-points attached on the brane. The interactions of string matter with the D-particle defects may not be universal among particle species. Due to electric charge conservation, electrically charged states, such as quarks or electrons may undergo only smooth ordinary scattering (recoil), while electrically neutral states, such as photons or neutrinos, may undergo topologically non trivial interactions in which a string state is split into two, with (some of) the corresponding end points are attached to the D-particle (“capture”). Capture includes recoil of the defect, as well as the induced back reaction onto space time, determined by means of conformal field theory methods of the associated non-critical string corresponding to the capture process [3]. For heavy (Galilean) D-particles, the analysis leads to induced metric deformations of the (initially) Minkowski, say, flat background of the form:

$$g_{00} = -1, g_{ij} = \delta_{ij}, \quad g_{i0} = g_{0i} = \frac{v_i}{c} \quad (1.1)$$

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where i, j spatial (four-dimensional) indices on the brane world, with 0 denoting a temporal direction, and v_i/c is the recoil 3-velocity of the D-particle defect, in units of the speed of light in vacuum, c , which by means of momentum conservation during the scattering process equals

$$v_i = g_s \Delta k_i / M_s . \quad (1.2)$$

In the last relation, Δk_i denotes the pertinent momentum transfer of matter, and M_s/g_s is the D-particle mass, with M_s the string scale and $g_s < 1$ the (weak) string coupling [3, 4]. We stress once more that the form (1.1) is valid for Galilean (heavy) D-particles, relative to the momenta of the incident matter states. In general, for relativistic D-particles there are higher order corrections of v_i in the expression for the recoil-induced metric, leading to a covariant form in terms of the *four-velocity* $u_\mu = \gamma(1, \vec{v})$, with γ the appropriate Lorentz factor [5]:

$$g_{\mu\nu} = \eta_{\mu\nu} + f(\Phi) u_\mu u_\nu \quad (1.3)$$

where the scalar factor $f(\Phi)$ depends on the details of the scalar dilaton Φ configuration in the model. For instance, in the impulse approximation to D-particle recoil, the Φ field is proportional to $u_\mu X^\mu$, as dictated by (logarithmic) conformal field theory considerations [5], and $f(\Phi) = \Theta_\epsilon(u_\mu X^\mu)$, with $\epsilon \rightarrow 0^+$ an appropriate regulator, linked to the world-sheet scale on account of closure of the conformal algebra [4]. However, more general forms for the function $f(\Phi)$ are possible, depending on the model.

As discussed in [2], the above mentioned capture process may entail flavour oscillation as well as a non-trivial Fock vacuum structure of the vacuum experienced by the flavour states instead of the corresponding mass eigenstates. When considering statistical distributions of particles, one may have on average $\langle u_i \rangle = 0$, with only non trivial stochastic fluctuations $\langle u_i u_j \rangle = \sigma^2 \delta_{ij}$, $\sigma^2 \neq 0$. The distribution of such D-particle velocities is a model dependent concept. In view of (1.1), a stochastic statistical fluctuation of u_i will also result in a stochastic fluctuation of the induced metric, and thus a stochastically fluctuating gravity theory.

In [2] we have studied the implications of such stochastic metrics in the simplified (but not unrealistic) case where the D-particle recoil velocities where along the direction of motion of the particle probe, thereby leading to an effectively two space-time dimensional problem, in which, however, the γ -matrix structures associated with the flavoured fermions (neutrinos) were kept four dimensional. For definiteness, in that work we have considered a model of Gaussian fluctuations, and we demonstrated that the associated oscillation probabilities of both fermion and boson flavoured particles exhibited exponential damping and decoherence, with the damping exponent scaling with the square of the time variable t .

It is the purpose of this work to extend these results to more generic models of stochastically fluctuating metrics, living in a full-fledged four dimensional space time. In particular, we assume small random fluctuations $h^{\mu\nu}$, around the Minkowski background inverse metric $\eta^{\mu\nu}$,

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \quad (1.4)$$

where for $\eta_{\mu\nu}$ we assume the sign convention $(1, -1, -1, -1)$ and $h^{\mu\nu}$ are defined as stochastic variables, obeying some statistical distribution, defining the model, such that $\langle h_{\mu\nu} \rangle = 0$, but $\langle h_{\mu\nu} h_{\rho\sigma} \rangle$ are non trivial.

The effect of neutrino oscillations will be studied in the above mentioned stochastic quantum gravity environment (1.4). Measurable quantities, such as transition amplitudes between neutrino flavours, will be obtained by averaging over the stochastic variables $h^{\mu\nu}$. As we will see, the influence of the quantum gravity environment has a decoherence effect on the neutrino states, which is equivalent to a damping of the oscillations between neutrino flavours.

The determination of the particular statistical distribution for the random fluctuations $h^{\mu\nu}$ requires complete knowledge of the quantum gravity theory. In the absence of such a theory, the statistical distribution can not be fixed. Hence, our considerations below will be largely phenomenological, in an attempt to impose bounds on the relevant parameters by comparison to experiment. To this end, we mention that the allowed distributions must be such that the corresponding characteristic function yields finite, well-defined results for physical observables, such as the flavour oscillation probability. This criterion allows the use of distributions, such as the Cauchy-Lorentz (Breit-Wigner resonance type), for which the mean (or odd moments) may not be well defined, while their variance (and higher even moments) diverge by themselves. For other distributions, such as the Gaussian [2], of course the latter entities are well defined and small. It is these two kinds of distributions for the metric fluctuations that we shall concentrate upon in this work. Within the context of our D-particle foam model [3], such distributions may characterise the ensembles of the D-particle recoil velocities. As we will see in the following sections, the choice of the distribution leads to different damping in the oscillation probability. For instance, the Cauchy-Lorentz case is characterised by exponential damping, with a linear power of time t , as in the Lindblad decoherence model [6, 7], in contrast to the Gaussian model [2], where the damping exponents are proportional to t^2 .

In this work we shall be only concerned with stochastic gravity models, with fluctuations (1.4) about flat Minkowski space-time backgrounds. In such models we saw that the results for the flavour oscillation probabilities are identical

between fermions and bosons. The situation may be different, if the background space times are not Minkowski flat. We shall come back to such more general situations in a forthcoming publication. Comparison of these two kinds, insofar as the order of magnitude of the expected effects, and their prospects of discovery in future experimental facilities, will be discussed in the concluding section of our article.

The structure of the article is as follows: in section II we set up the formalism for scalar flavoured particles, concentrating on the case of two generations for simplicity. We examine first the situation, in which only leading order (linear) terms in the deformation $h_{\mu\nu}$ are kept in the expressions for the pertinent oscillation probabilities, paying attention to demonstrate the dependence on the results, in particular the form of the damping exponents as functions of time, on the kind of the (statistical) distribution of the metric fluctuations. We also demonstrate that the inclusion of higher order corrections in $h_{\mu\nu}$ do not change the results qualitatively. In section III we extend the analysis to two-generations of fermions, showing that the results concerning the oscillation probabilities and the effects of decoherence due to the stochastic metric fluctuations, are identical to the scalar case. Finally we comment briefly on a possible extension of the above results to the realistic three generation neutrino case, which however shall be considered in detail in a forthcoming publication. Conclusions and outlook are presented in section IV.

II. FLAVOUR OSCILLATIONS FOR SCALAR PARTICLES IN A STOCHASTIC METRIC BACKGROUND

In this section we study massive scalar particles in the quantum gravity stochastic background (1.4). The Klein-Gordon equation for a single scalar particle in a gravitational background, reads:

$$(g^{\alpha\beta}\nabla_\alpha\nabla_\beta + m^2)\phi = 0 \quad (2.1)$$

where $g^{\alpha\beta}$ is the inverse metric tensor and ∇_α is a gravitational covariant derivative.

A flat fluctuation $h^{\mu\nu}$ around the inverse Minkowski metric $\eta^{\mu\nu}$ is defined by Eq. (1.4). For simplicity, and motivated by the Galilean D-particle foam case (1.1), we may assumed that $h^{\mu\nu}$ is independent of space-time coordinates [17]. In such a case, the Christoffel symbols vanish, hence the Klein-Gordon equation in 3+1 dimensions reads:

$$(g^{00}\partial_0\partial_0 + 2g^{0i}\partial_0\partial_i + g^{ij}\partial_i\partial_j + m^2)\phi = 0 \quad (2.2)$$

We seek for plane wave solutions of the form:

$$\phi(x, t) = \hat{\phi}(k, \omega)e^{i(\omega t - kx)} \quad (2.3)$$

where k represents the momentum of the scalar particle in the Minkowski metric background. Note, that we have chosen one of the coordinate axes to lie along the direction of motion of the scalar particles, i.e. we have set $k_2 = k_3 = 0$ and $k_1 = k$. Plugging the expression (2.3) into eq.(2.1) leads to

$$g^{00}\omega^2 - 2g^{01}k\omega + g^{11}k^2 - m^2 = 0 \quad (2.4)$$

The two solutions $\omega(k)$, $\omega'(k)$ of the above equation are physically equivalent, since

$$\omega(-k) = -\omega'(k) . \quad (2.5)$$

Keeping the positive energy solution, we obtain the dispersion relation

$$\omega = \frac{g^{01}}{g^{00}}k + \frac{1}{g^{00}}\sqrt{(g^{01})^2k^2 - g^{00}(g^{11}k^2 - m^2)} \quad (2.6)$$

We next proceed to discuss the flavour oscillations, by considering first the approximation of keeping at most linear terms in the stochastic fluctuations $h_{\mu\nu}$.

A. Linear approximation for the stochastic metric fluctuations

We assume for brevity two flavours for the scalar particles, written as a two-component scalar $\Phi = (\phi_1, \phi_2)$. The equation of motion is

$$(g^{\alpha\beta}\nabla_\alpha\nabla_\beta + M^2)\Phi = 0 \quad (2.7)$$

where M^2 is the 2×2 positive definite (mass)² matrix, with eigenvalues m_1^2 and m_2^2 . Note, that the flavour states are not eigenstates of M . For $t = 0$, we assume the production of a scalar particle of flavour α with density matrix:

$$\rho(0) = |\phi_\alpha\rangle\langle\phi_\alpha| \quad (2.8)$$

Taking into account that the energy eigenstates $|f_l\rangle$ ($l = 1, 2$) are related to the flavour states $|\phi_\alpha\rangle$ via the unitary transformation:

$$|\phi_\alpha\rangle = \sum_i U_{\alpha i} |f_i\rangle \quad \langle\phi_\alpha| = \sum_j U_{\alpha j}^* \langle f_j| \quad (2.9)$$

$$|f_i\rangle = \sum_\beta U_{\beta i}^* |\phi_\beta\rangle \quad \langle f_j| = \sum_\gamma U_{\gamma j} \langle\phi_\gamma| \quad (2.10)$$

we then obtain from Eqs. (2.8) and (2.9):

$$\rho(0) = \sum_{i,j} U_{\alpha i} U_{\alpha j}^* |f_i\rangle\langle f_j| \quad (2.11)$$

where we sum only over i, j , with α denoting the initial flavour state. The time evolution of the density matrix yields:

$$\rho(t) = \sum_{i,j} U_{\alpha i} U_{\alpha j}^* e^{i(\omega_i - \omega_j)t} |f_i\rangle\langle f_j| \quad (2.12)$$

Upon using Eq. (2.10), one obtains the density matrix at time t , expressed in the basis of flavour states:

$$\rho(t) = \sum_{i,j,\beta,\gamma} U_{\alpha i} U_{\alpha j}^* U_{\beta i}^* U_{\gamma j} e^{i(\omega_i - \omega_j)t} |\phi_\beta\rangle\langle\phi_\gamma|. \quad (2.13)$$

The probability to find the scalar particle in a state of flavour β ($\beta \neq \alpha$) is

$$\text{Prob}(\alpha \rightarrow \beta) = \sum_{i,j} U_{\alpha i} U_{\alpha j}^* U_{\beta i}^* U_{\beta j} e^{i(\omega_i - \omega_j)t} \quad (2.14)$$

where the time dependent part is

$$U_{\alpha 1} U_{\alpha 2}^* U_{\beta 1}^* U_{\beta 2} e^{i(\omega_1 - \omega_2)t} + U_{\alpha 2} U_{\alpha 1}^* U_{\beta 2}^* U_{\beta 1} e^{i(\omega_2 - \omega_1)t} \quad (2.15)$$

Since the perturbations $h_{\mu\nu}$ are stochastic variables, it is necessary to compute the following integral

$$\langle e^{i(\omega_1 - \omega_2)t} \rangle = \int d^{10}\mathbf{h} F(\mathbf{h}) e^{i(\omega_1 - \omega_2)t} \quad (2.16)$$

where $\mathbf{h} = (h^{00}, h^{01}, h^{02}, \dots, h^{23}, h^{33})$, and $F(\mathbf{h})$ is a multi-variable probability density function. The measurable transition probability is found by averaging over the metrics

$$\langle \text{Prob}(\alpha \rightarrow \beta) \rangle = \sum_{i,j} U_{\alpha i} U_{\alpha j}^* U_{\beta i}^* U_{\beta j} \langle e^{i(\omega_i - \omega_j)t} \rangle \quad (2.17)$$

In what follows we will assume that the stochastic variables $h^{\mu\nu}$ are independent, or equivalently that:

$$F(\mathbf{h}) = f(h^{00})f(h^{01})f(h^{02}) \times \dots \times f(h^{23})f(h^{33}) \quad (2.18)$$

where $f(x)$ is a one-variable probability density function for the stochastic variable x . Additionally, we have assumed that the one-variable density function $f(x)$ is the same for all $h^{\mu\nu}$. We will approximate the energy difference $\omega_1 - \omega_2$ with a linear expansion of Eq. (2.6) over the fluctuations \mathbf{h}

$$\omega_1(\mathbf{h}) - \omega_2(\mathbf{h}) = a - \frac{a}{2}h^{00} - \frac{b}{2}h^{11} + O(\mathbf{h}^2) \quad (2.19)$$

where

$$a = \sqrt{k^2 + m_1^2} - \sqrt{k^2 + m_2^2}, \quad b = \frac{k^2}{\sqrt{k^2 + m_1^2}} - \frac{k^2}{\sqrt{k^2 + m_2^2}} \quad (2.20)$$

Note that in the above linear expansion only the elements h^{00} and h^{11} are involved. The integral (2.16) then reduces to

$$\begin{aligned}\langle e^{i(\omega_1 - \omega_2)t} \rangle &= \int dh^{00} dh^{11} f(h^{00}) f(h^{11}) \exp \{ it (a - ah^{00}/2 - bh^{11}/2) \} \\ &= \Phi(-at/2) \Phi(-bt/2) \exp(iat)\end{aligned}\quad (2.21)$$

where

$$\Phi(\xi) = \int_{-\infty}^{+\infty} e^{i\xi x} f(x) dx \quad (2.22)$$

is the characteristic function of the stochastic variable x with probability density function $f(x)$.

B. Gaussian fluctuations

We will assume first a Gaussian distribution for the metric fluctuations, with density function

$$f(x) = \frac{e^{-x^2/\sigma^2}}{\sqrt{\pi\sigma^2}} \quad (2.23)$$

where we have considered the case of zero mean value ($\mu = 0$), and σ is the corresponding standard deviation. The characteristic function for the Gaussian distribution is:

$$\Phi(\xi) = \frac{1}{\sqrt{\pi\sigma^2}} \int_{-\infty}^{+\infty} dx \exp(i\xi x - x^2/\sigma^2) = \exp(-\xi^2\sigma^2/4) \quad (2.24)$$

From Eq. (2.21) we obtain

$$\langle e^{i(\omega_1 - \omega_2)t} \rangle = \exp \left\{ iat - \frac{\sigma^2 t^2}{16} (a^2 + b^2) \right\} \quad (2.25)$$

Hence, the amplitude of the flavour oscillations has an exponential damping quadratic in time, that depends on the particle's momentum. In the case of high energy particles (compared to their mass) one can make an asymptotic expansion for $m_i/k \ll 1$

$$\langle e^{i(\omega_1 - \omega_2)t} \rangle \simeq \exp \left\{ ikt\Delta \left(1 - \frac{m_1^2 + m_2^2}{4k^2} \right) \right\} \exp \left\{ -\frac{\sigma^2 (kt)^2}{8} \Delta^2 \right\} \quad (2.26)$$

where terms of order $\mathcal{O}(m_i/k)^6$ have been disregarded, and the parameter Δ is defined as

$$\Delta = \frac{m_1^2 - m_2^2}{2k^2} \ll 1. \quad (2.27)$$

We remark at this point that self-consistency in the expansion in powers of m_i/k forces us to keep the forth-order correction term in the expression for the oscillation frequency in (2.26), since the damping is of order $(m_i/k)^4$. The results are in full agreement with the corresponding effective two-dimensional case of [2]. A discussion on the prospects of observing such a decoherence effect in realistic physical systems of neutrinos, as well as on the dependence of the effect on the distance of the neutrino sources, especially in the case of astrophysical neutrinos, will be given in the concluding section of the article.

C. Cauchy-Lorentz fluctuations

We will consider next the case of Cauchy-Lorentz fluctuations

$$f(x) = \frac{1}{\pi} \frac{\gamma}{x^2 + \gamma^2} \quad (2.28)$$

where γ is the scale parameter, and the location parameter is zero. Note that the Cauchy-Lorentz distribution has no well defined mean, and in general odd higher-order moments, while its variance and higher-order even moments diverge. However, the characteristic function, defined as:

$$\Phi(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \frac{\gamma e^{i\xi x}}{x^2 + \gamma^2} = \exp(-\gamma|\xi|) \quad (2.29)$$

yields finite results for the oscillation probabilities:

$$\langle e^{i(\omega_1 - \omega_2)t} \rangle = \exp \left\{ iat - \frac{\gamma t}{2} (|a| + |b|) \right\} \quad (2.30)$$

In this situation we obtain an exponential damping, with the exponent linear in time. An expansion in powers of m_i/k gives

$$\langle e^{i(\omega_1 - \omega_2)t} \rangle \simeq \exp \left\{ ikt\Delta - \gamma kt|\Delta| \right\}. \quad (2.31)$$

where terms of order $\mathcal{O}(m_i/k)^4$ have been disregarded. We did not write here the correction to the oscillation frequency, as this is of order $(m_i/k)^4$, whereas the damping is of order $(m_i/k)^2$, and we are interested in the dominant effect only. The case is similar in form to the Lindblad decoherence [6, 7].

D. Beyond the linear approximation

In this section we go beyond the linear approximation for Gaussian metric fluctuations, in order to check whether the previously derived damping of flavour oscillations in the linear- h approximation is modified significantly. As we shall see, the main results remain valid. In this case we expand the energy difference $\omega_1 - \omega_2$ up to quadratic terms in the fluctuations \mathbf{h}

$$\omega_1(\mathbf{h}) - \omega_2(\mathbf{h}) = a + \mathbf{h} \cdot \mathbf{d} + \mathbf{h} \cdot \mathbf{D} \cdot \mathbf{h} + \mathcal{O}(\mathbf{h}^3) \quad (2.32)$$

with $\mathbf{h} = (h^{00}, h^{01}, h^{11})$, $\mathbf{d} = (-a/2, 0, -b/2)$ and

$$\mathbf{D} = \begin{pmatrix} 3a/4 & 0 & b/4 \\ 0 & b & 0 \\ b/4 & 0 & -c/4 \end{pmatrix} \quad (2.33)$$

The parameters a and b are given by Eq. (2.20), while c is defined as:

$$c = \frac{k^4}{k^2 + m_1^2} - \frac{k^4}{k^2 + m_2^2} \quad (2.34)$$

We next write the Gaussian probability density for the metric fluctuations in a compact form:

$$F(\mathbf{h}) = \frac{e^{-\mathbf{h} \cdot \mathbf{\Xi} \cdot \mathbf{h}}}{(\sqrt{\pi}\sigma)^3} \quad (2.35)$$

where

$$\mathbf{\Xi} = \text{diag} \left(\frac{1}{\sigma^2}, \frac{1}{\sigma^2}, \frac{1}{\sigma^2} \right) \quad (2.36)$$

We wish to compute the integral

$$\langle e^{i(\omega_1 - \omega_2)t} \rangle = \int d^3\mathbf{h} \frac{e^{-\mathbf{h} \cdot \mathbf{\Xi} \cdot \mathbf{h}}}{(\sqrt{\pi}\sigma)^3} e^{i(\omega_1 - \omega_2)t} \quad (2.37)$$

using the formula

$$\int d^3\mathbf{h} \exp(-\mathbf{h} \cdot \mathbf{B} \cdot \mathbf{h} + \mathbf{u} \cdot \mathbf{h}) = \frac{\pi^{3/2}}{\det \mathbf{B}} \exp \left(\frac{\mathbf{u} \cdot \mathbf{B}^{-1} \cdot \mathbf{u}}{4} \right) \quad (2.38)$$

where the matrices \mathbf{B} and \mathbf{u} are defined as:

$$\mathbf{B} = \mathbf{\Xi} - it\mathbf{D} \quad (2.39)$$

$$\mathbf{u} = it\mathbf{d} \quad (2.40)$$

The matrix \mathbf{B} can be written explicitly:

$$\mathbf{B} = \begin{pmatrix} \frac{1}{\sigma^2} - \frac{3}{4}ita & 0 & -\frac{1}{4}itb \\ 0 & \frac{1}{\sigma^2} - itb & 0 \\ -\frac{1}{4}itb & 0 & \frac{1}{\sigma^2} + \frac{1}{4}itc \end{pmatrix} \quad (2.41)$$

Upon applying Eq. (2.38), we find:

$$\begin{aligned} \langle e^{i(\omega_1 - \omega_2)t} \rangle &= \left(\frac{\det \mathbf{\Xi}}{\det \mathbf{B}} \right)^{1/2} \exp(-\chi(t)) \exp(iat) \\ &= \frac{4}{\sqrt{P(t)}} \exp(-\chi(t)) \exp(iat) \end{aligned} \quad (2.42)$$

where

$$\begin{aligned} \chi(t) &= \frac{\sigma^2 t^2 (4(a^2 + b^2) - 4i\sigma^2 t(b^2 - ac))}{4\sigma^4 t^2(b^2 + 3ac) - 16i\sigma^2 t(3a - c) + 64} \\ P(t) &= (1 - ib\sigma^2 t)(16 - 4i(3a - c)\sigma^2 t + (b^2 + 3ac)\sigma^4 t^2) \end{aligned} \quad (2.43)$$

Expanding the exponent $\chi(t)$ in powers of the small parameter σ we obtain

$$\begin{aligned} \chi(t) &= \frac{\sigma^2 t^2}{16}(a^2 + b^2) + \frac{i\sigma^4 t^3}{64}(3a^3 + 2ab^2 - b^2 c) \\ &\quad - \frac{\sigma^6 t^4}{256}(9a^4 + 7a^2 b^2 + b^4 - 2ab^2 c + b^2 c^2) + O(\sigma^8) \end{aligned} \quad (2.44)$$

The leading term of the above expansion corresponds to the main effect which is the exponential damping of particle oscillations. Note that it is identical with the one found in the linear approximation above. The next to leading order term is purely imaginary and modifies slightly the oscillation term e^{iat} . The factor $P(t)$ has a subleading contribution of the form:

$$|P(t)| = 16 + \frac{\sigma^4 t^2}{2}(9a^2 + 18b^2 + c^2) + O(\sigma^6) \quad (2.45)$$

The reader should compare the results with the effectively two-dimensional model of [2]. The results are similar, independently of the representation of gravity fluctuations and the number of spatial dimensions.

III. FERMIONS IN A STOCHASTIC METRIC BACKGROUND

In this section we would like to extend the above results to incorporate particles with spin (fermions), which from a phenomenological point of view is more interesting, in view of the potential application to the physics of neutrino oscillations. However, as we shall demonstrate below, the results will be similar to the bosonic case, as far as the main features of decoherence damping is concerned. This is attributed to expanding about a flat Minkowski background (1.4).

A. Dirac fermion dispersion relation

In this section, we will consider Dirac fermions, but the results presented are the same for Majorana fermions, as the only difference is that the latter are described by self-conjugate fields, which does not affect the dispersion relation. We remark that this difference would play a rôle only if we considered the MSW effect [8], for the propagation of neutrinos in matter media, as in that case only one Weyl spinor component of the fermion would be affected. We shall

consider this case, along with more general situations involving expansions about non flat backgrounds in a future publication.

We review here the basic elements we will need to describe fermions in a curved background. We shall follow the formalism of [9] and in references therein, where we refer the reader for details. At each point \mathbf{M} of space time, the vierbeins $\mathbf{e}_\alpha = \partial_\alpha \mathbf{M}$ span the flat tangent space time, and are related to the inverse metric by

$$e_\alpha^\mu e_\beta^\nu \eta^{\alpha\beta} = g^{\mu\nu} \quad (3.1)$$

The gamma matrices γ^α which follow are defined on the flat tangent space time and satisfy $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$, and we also define $\sigma^{\alpha\beta} = i[\gamma^\alpha, \gamma^\beta]/2$.

The Dirac equation in a curved background is

$$(i\gamma^\alpha \mathcal{D}_\alpha - m) \psi = 0 \quad (3.2)$$

where

$$\mathcal{D}_\alpha = e_\alpha^\mu \left[\partial_\mu - \frac{i}{4} e_\beta^\nu \nabla_\mu e_{\rho\nu} \sigma^{\beta\rho} \right] \quad (3.3)$$

and ∇_μ is the covariant derivative. In the present situation, we are interested in a constant and homogeneous stochastic metric, such that the vierbeins do not depend on space time coordinates, and the Christoffel symbols vanish. As a consequence, the Dirac equation reads

$$(i\gamma^\alpha e_\alpha^\mu \partial_\mu - m) \psi = 0 \quad (3.4)$$

In order to find the dispersion relation for the fermion, we multiply the Dirac equation (3.4) by the complex conjugate operator $(-i\gamma^\beta e_\beta^\nu \partial_\nu - m)$ to obtain

$$\begin{aligned} 0 &= \left(\frac{1}{2} \{\gamma^\alpha, \gamma^\beta\} e_\alpha^\mu e_\beta^\nu \partial_\mu \partial_\nu + m^2 \right) \psi \\ &= (g^{\mu\nu} \partial_\mu \partial_\nu + m^2) \psi \end{aligned} \quad (3.5)$$

which is similar to the equation (2.1) for a boson, since covariant derivatives are simple derivatives in our case. As a consequence, the spin does not play a role in the situation where the background metric is flat, and the previous results derived for a boson apply to fermions. It appears that damping is a general result of quantum gravity fluctuations.

B. Two-flavour fermion oscillations

We consider two Dirac fermions ψ_e, ψ_μ , written as an eight-component fermion Ψ , which are coupled by a Dirac mixing mass matrix, and are described by the equation of motion

$$(i\gamma^\alpha \mathcal{D}_\alpha - M) \Psi = 0, \quad (3.6)$$

where the mass matrix in flavour space reads:

$$M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix} \quad (3.7)$$

In order to involve the mass eigenstates ψ_1, ψ_2 , one performs the following rotation in flavour space [10]

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (3.8)$$

where

$$\tan(2\theta) = \frac{2m_{e\mu}}{m_\mu - m_e} \quad (3.9)$$

As explained in [10], the Lagrangian describing the fermions ψ_1 and ψ_2 is then the sum of two independent Lagrangians, such that the equations of motion for ψ_1 and ψ_2 are

$$(i\gamma^\alpha \mathcal{D}_\alpha - m_j) \psi_j = 0 \quad j = 1, 2, \quad (3.10)$$

where $m_1 + m_2 = m_e + m_\mu$ and $m_1 m_2 = m_e m_\mu - m_{e\mu}^2$. Hence, this leads to the study of two individual fermions. The results are then similar to those derived for bosons, as the fermion dispersion relation is identical with that of bosons, as can be readily seen from Eq. (3.5).

Thus, using Eq. (2.14) and Eqs. (2.25), (2.31) and (3.8), one obtains the formula for the transition probability between flavours, in the linear-order approximation for the \hbar -metric fluctuations:

$$\langle \text{Prob}(\alpha \rightarrow \beta) \rangle = \frac{1}{2} \sin^2(2\theta) \left(1 - e^{-\chi(t)} \cos(at) \right), \quad (3.11)$$

where $\alpha \neq \beta$ and

- $\chi(t) = \frac{\sigma^2 t^2}{16} (a^2 + b^2)$ for the Gaussian case;
- $\chi(t) = \frac{\gamma t}{2} (|a| + |b|)$ for the Cauchy-Lorentz case.

We note that, in the case where there is no metric fluctuation (for $\chi(t) = 0$), the probability (3.11) leads to the known result $\frac{1}{2} \sin^2(2\theta)[1 - \cos(at)]$ [11]. Taking into account metric fluctuations, we see that the limit $t \rightarrow \infty$ leads to the stationary probability $\frac{1}{2} \sin^2(2\theta)$, which is independent of the fermion energy.

C. Three-flavour fermion oscillations

It is not difficult to generalize our results in the case of three flavours. As we will see the exponential damping remains.

The corresponding 3×3 Dirac mass matrix M , defined in analogy with Eq. (3.7), can be diagonalized with a unitary transformation $U_{\alpha i}$:

$$|\psi_\alpha\rangle = \sum_i U_{\alpha i} |\psi_i\rangle \quad (\alpha = e, \mu, \tau) \quad (3.12)$$

and has three positive eigenvalues m_i ($i = 1, 2, 3$). We note for completeness that for the U matrix we use the parametrization of [11].

The probability to find a fermion, with initial state α , in a final state with flavour $\beta \neq \alpha$ is given by Eq. (2.17):

$$\langle \text{Prob}(\alpha \rightarrow \beta) \rangle = \sum_{i,j} U_{\alpha i} U_{\alpha j}^* U_{\beta i} U_{\beta j} \langle e^{i(\omega_i - \omega_j)t} \rangle. \quad (3.13)$$

As in the two-flavour case, the probability (3.13) has a time-independent part, which is

$$|U_{\alpha 1}|^2 |U_{\beta 1}|^2 + |U_{\alpha 2}|^2 |U_{\beta 2}|^2 + |U_{\alpha 3}|^2 |U_{\beta 3}|^2, \quad (3.14)$$

whereas its time-dependend part is

$$U_{\alpha 1} U_{\alpha 2}^* U_{\beta 1}^* U_{\beta 2} \langle e^{i(\omega_1 - \omega_2)t} \rangle + U_{\alpha 2} U_{\alpha 3}^* U_{\beta 2}^* U_{\beta 3} \langle e^{i(\omega_2 - \omega_3)t} \rangle + U_{\alpha 3} U_{\alpha 1}^* U_{\beta 3}^* U_{\beta 1} \langle e^{i(\omega_3 - \omega_1)t} \rangle + cc. \quad (3.15)$$

We then average over the stochastic fluctuations $h^{\mu\nu}$, and calculate the three elements

$$\langle e^{i(\omega_1 - \omega_2)t} \rangle, \quad \langle e^{i(\omega_2 - \omega_3)t} \rangle, \quad \langle e^{i(\omega_3 - \omega_1)t} \rangle.$$

Using our previous results (see Eqs. (2.26) and (2.31)), we obtain for Gaussian fluctuations

$$\langle e^{i(\omega_i - \omega_j)t} \rangle \simeq \exp \left\{ ikt \Delta_{ij} \left(1 - \frac{m_i^2 + m_j^2}{4k^2} \right) \right\} \exp \left\{ -\frac{\sigma^2 (kt)^2}{8} \Delta_{ij}^2 \right\}, \quad (3.16)$$

and for Cauchy-Lorentz fluctuations

$$\langle e^{i(\omega_i - \omega_j)t} \rangle \simeq \exp \left\{ ikt \Delta_{ij} - \gamma kt |\Delta_{ij}| \right\}, \quad (3.17)$$

where

$$\Delta_{ij} = \frac{m_i^2 - m_j^2}{2k^2} < 1. \quad (3.18)$$

The reader is reminded at this point that, as with (2.26), consistency in the expansion in powers of m_i/k forces us to keep the forth-order correction in m_i/k in the oscillation frequency (3.16) of the Gaussian model, given that the damping is of order $(m_i/k)^4$. On the other hand, in the Cauchy-Lorentz case (3.17) we disregarded such a forth order correction, since the corresponding damping is of order $(m_i/k)^2$.

We now notice that the $1/E$ -dependent exponential damping in the case (3.17) may be translated [13, 14] as implying a finite life time τ_{lab} for the probe in the laboratory frame:

$$\exp\left(-t \frac{\gamma|m_i^2 - m_j^2|}{2E}\right) \equiv \exp\left(-\frac{t}{\tau_{\text{lab}}}\right) = \exp\left(-\frac{tm_{\nu_i}}{E_{\nu_i}\tau_{\nu_i}}\right) \quad (3.19)$$

where τ_{ν_i} is the life-time of the neutrino probe in the rest frame of the massive neutrino. It is interesting therefore to use our microscopic models in order to identify a possible quantum-gravitational origin of a finite lifetime of neutrinos. We shall come back briefly to this point later on the article, when we discuss lower bounds on the decoherence-induced neutrino lifetime (3.19) from (current and future) experiments.

IV. CONCLUSIONS AND OUTLOOK

In this concluding section we would like to discuss briefly the above results in light of the prospects for observing the above effects in current or upcoming neutrino facilities. A more detailed comparison with experimental data and derivation of bounds will appear in a forthcoming publication. The main results of our work was the exponential decoherence-induced damping in the oscillation probability, which for the case of Gaussian fluctuations, with variance σ^2 , is given by (3.16), (3.18), while for the case of Cauchy-Lorentz distribution by (3.17).

For realistic models of quantum gravity, we first observe that in the Gaussian model, the damping exponent is much more suppressed than the corresponding one in the Cauchy-Lorentz (Lindblad-like-time-scaling) case. This is due to the extra suppression factor $(m_1^2 - m_2^2)/E$ appearing in the exponent of the Gaussian model. In order to make direct comparison with experiment, it is essential to use as concrete quantum gravity models as possible. To this end, the D-particle foam model [3] turns out to be very useful. According to this model, the stochastic metric fluctuations are *induced*, as explained in the introduction, by means of the recoil of the D-particle defect during its topologically non-trivial interaction/capture process with matter. Considering a gas of D-particles, it is natural to consider a Gaussian model for the distribution of the respective (spatial) recoil velocities, u_i , which in turn induce the metric fluctuations (1.1), for heavy defects. In such a case, the distribution variables are the dimensionless ratios u_i/c , which however are restricted to be less than one in magnitude, as u_i are not allowed to exceed the speed of light in vacuum c , as requested by string theory. In such a case, the relevant integrals in the previous sections, e.g. (2.24), have to be understood to be restricted to the range $[-1, 1]$.

In this sense, the above results can only be viewed as an idealisation of the situation characterising the D-foam model, which however, is a pretty good approximation of reality if the variance of the Gaussian distribution is pretty small, which is the case in the D-foam model. Indeed, taking into account that σ^2 in such a case might *naturally* be expected to have an order of magnitude dictated by the square of (1.2), since after all σ^2 describes the variance of D-particle recoil velocities, which was assumed small, and the typical order of the latter is given by (1.2). If one assumes that the momentum transfer of a particle probe, during its capture by a recoiling D-particle spacetime defect, is of the order of the particles momentum itself, then we may have for the variance σ^2 the estimate:

$$\sigma^2 = \mathcal{O}\left(\frac{E^2}{M_s^2 g_s^2}\right) \quad (4.1)$$

This yields a damping exponent (3.16) of order (in units $\hbar = c = 1$):

$$\exp(-\Omega_{\text{Gauss}}^2 t^2) \ , \quad \Omega_{\text{Gauss}}^2 = \frac{g_s^2(m_i^2 - m_j^2)^2}{32 M_s^2} \ . \quad (4.2)$$

For the case of Cauchy Lorentz distribution of D-particle velocities (adopted appropriately, as in the Gaussian case, to incorporate velocity variables u_i which do not exceed the speed of light in vacuum c), one observes that the parameter γ provides a characteristic scale for D-particle velocities, since, if they exceed that scale the distribution is diminished significantly. On account of (1.2), this leads to the assumption that a natural order of magnitude for γ , in the context of the D-particle foam model, is $\gamma \sim g_s E/M_s$, with E the energy of the particle probe. The corresponding decoherence-induced damping exponent has then the form:

$$\exp(-\Omega_{\text{CL}} t) \ , \quad \Omega_{\text{CL}} = \frac{g_s|m_i^2 - m_j^2|}{2 M_s} \ . \quad (4.3)$$

Searches for Lindblad decoherence [12], using the latest neutrino experimental data, have bounded the respective coefficients in a stringent way. For two-flavour models, the parametrization used in [12] for the decoherence-induced Lindblad-type damping coefficients is:

$$\exp(-\tilde{\gamma} t) , \quad \tilde{\gamma} = \gamma_0 \left(\frac{E}{\text{GeV}} \right)^n \quad (4.4)$$

with the following bounds provided by means of combining atmospheric, solar-neutrino and KamLand data [12]

$$\begin{aligned} \gamma_0 &< 0.67 \times 10^{-24} \text{ GeV} , \quad n = 0 \\ \gamma_0 &< 0.47 \times 10^{-20} \text{ GeV} , \quad n = 2 \\ \gamma_0 &< 0.78 \times 10^{-26} \text{ GeV} , \quad n = -1 \end{aligned} \quad (4.5)$$

It should be remarked that all these bounds should be taken with a grain of salt, since there is no guarantee that in a theory of quantum gravity γ_0 should be the same in all channels, or that the functional dependence of the decoherence coefficients γ on the probe's energy E follows a simple power law. Complicated functional dependencies $\gamma(E)$ might be present in general.

To compare with our model above (4.3), we observe that the resulting damping coefficient is independent of the probe's energy E , as a result of the E -dependence of the coefficient γ . For such constant decoherence coefficients, the analysis of [12], (4.5), yields the following bound on M_s/g_s :

$$\frac{M_s}{g_s} > 0.74 \times 10^{24} \max_{i,j} \left[\left(\frac{|m_i^2 - m_j^2|}{\text{GeV}^2} \right) \text{ GeV} \right] \quad (4.6)$$

where $\max_{i,j}$ indicates the maximum mass difference among neutrino flavours, and we assumed that the quantum gravity parameters are the same for all flavours, which is certainly the case of the D-particle foam model. Recent data [12] indicate that $\Delta m_{ij}^2 \in (10^{-23} - 10^{-21}) \text{ GeV}^2$, from which (4.6) implies:

$$\frac{M_s}{g_s} > 740 \text{ GeV} \quad (4.7)$$

The expected minimal value of the string mass scale M_s is, of course at least a few TeV, for which the string coupling must be very weak in order to provide realistic string phenomenology. For couplings of order $g_s \leq 1/2$ the phenomenologically correct scale is close to four-dimensional Planck scale, $M_s \sim 10^{18} \text{ GeV}$. The above considerations, therefore, imply that the currently available neutrino data do not have the sensitivity to probe realistic D-particle foam models (the Gaussian Models decoherence is much more suppressed than the Cauchy-Lorentz one, as already mentioned).

One may reverse the logic, and try to consider bounds in as much model independent way as possible, using the results (3.16), (3.17), without reference to any explicit model for the parameters σ^2 and γ . In such a case, one may assume that these parameters are probe-energy independent, and try to bound their values, by comparing with data. We postpone such a complete analysis for a forthcoming work. Here, however, we note in brief that in the Cauchy-Lorentz case with constant scale-parameter γ , the resulting decoherence coefficient corresponds to the $1/E$ -dependent case, $n = -1$ in (4.4), for which, on account of (4.5), one obtains the following bound on γ :

$$\gamma < 10^{-5} \quad (4.8)$$

on account of the above-mentioned measured neutrino mass differences.

A final comment, concerns the order of the damping exponents in (3.16) and (3.17) for astrophysical neutrinos. To answer such a question it is imperative to know the energy dependence of the respective damping coefficient. Since the decoherence effects depend on the actual time of flight of neutrino t , the effects are maximised for extraterrestrial neutrinos, coming from extragalactic sources. Potential limits on Lindblad decoherence using high energy (TeV scale) astrophysical (anti)neutrinos from ICE-CUBE in the future have been analysed in [14] and also in [15]. In the ICE-CUBE case [14], the inverse energy $n = -1$ decoherence coefficient γ_0 in the terminology (4.4), is found to be (at a 90% CL) $\gamma_0 < 10^{-34} \text{ GeV}$, which will improve the existing sensitivity by eight orders of magnitude, implying, for a constant Cauchy-Lorentz distribution, the limit $\gamma < 10^{-13}$. Such sensitivities are not far from naturally expected values of γ in space-time foam models involving heavy (Planck-mass $M_s/g_s \sim 10^{19} \text{ GeV}$) D-particles and TeV-momentum transfers during the capture of TeV-energy (anti)neutrino matter by the recoiling D-particle. Moreover, for completeness, we mention that, as discussed in [14], the 90% CL bound on the inverse-energy decoherence from ICE-CUBE will imply,

according to (3.19) the existing bounds on the electron-antineutrino life time $\tau_{\bar{\nu}_e}/m_{\bar{\nu}_e} > 10^{34} \text{ GeV}^{-2}$, improving by four orders of magnitude the existing bounds from solar neutrinos.

Finally, we mention that in [16] it was argued that ultra-high energy neutrinos, with energies $10^{17} - 10^{19} \text{ eV}$ can be emitted by Gamma Ray Bursts (GRB), which actually carry a significant fraction of the GRB energy. Most of GRBs lie at cosmological distances corresponding to distances $z > 1$, i.e. larger than 10^{27} m . To study self-consistently such cases, one needs of course to consider the propagation of the neutrino in a Robertson-Walker background, about which one could expand the space-time metric fluctuations. We hope to come to a study of such issues in a future work. However, to get a preliminary idea it suffices to ignore the expansion of the Universe and consider the Cauchy-Lorentz decoherence distribution as a pilot case. For such distances and energies, the relevant exponent (3.17) of a Cauchy-Lorentz decoherence model with constant (probe-energy-independent) $\gamma < 10^{-5}$ (c.f. (4.8)), becomes roughly of order: $5 \times 10^{-37} \times t$, where we took into account that the maximal of the neutrino mass difference is $\max_{i,j} \Delta m_{ij} \sim 10^{-3}$. Such damping becomes of order one for distances of order $L \sim 10^{27} \text{ m}$, i.e. at cosmological distances of GRBs. Of course, the expansion of the Universe, will modify such results, nevertheless what this preliminary exercise showed is that high energy astrophysical neutrinos can indeed constitute sensitive probes of decoherence models [14, 15]. We hope to be able to come back to a detailed discussion of such issues in the future.

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